UNIVERSITA' DEGLI STUDI DI SIENA

DIPARTIMENTO DI STUDI AZIENDALI E SOCIALI
SEZIONE: "POPOLAZIONE, AMBIENTE E METODI MATEMATICO-STATISTICI"

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# A NEW FEMALE FERTILITY FUNCTION

**Quaderno n.2** **Dicembre 1996**

# A NEW FEMALE FERTILITY FUNCTION

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1.1 In the period of female fertile life the relative fertility may be considered from two points of view: a theoretical one corresponding to the maximum number of children reproducible compatibly with physiological capacity, and the other referring to real reproduction.

In the first case, excluding multiple births, it can be said that the physiological capacity of a woman is high at the beginning of fertile life and then decreases until it dies out at around 50–55 years of age. In reality however this capacity is not completely expressed but is modified by various factors.

1.2 Physiological capacity, that is potential fertility, may be theoretically represented by the descending branch in the first quadrant of an ellipses represented by:

$$f^F(X) = \sqrt{a^2\left(1 - X^2/b^2\right)} \qquad [1]$$

As, generally speaking, fertility rates begin to have significative values in age U and finish in age V, it is necessary to perform a variable transformation of value U so that the theoretical fertility distribution extends from age zero to age V–U in accordance with the branch of the studied ellipse. The general relation then is:

$$f^F(X) = \sqrt{a^2\left[1 - (X-U)^2/b^2\right]}. \qquad [2]$$

In two extreme points of fertility we have:

$$f^F(U) = \sqrt{a^2\left[1 - (U-U)^2/b^2\right]}.=1 \qquad [3]$$

$$f^F(V) = \sqrt{a^2\left[1 - (V-U)^2/b^2\right]}.=0$$

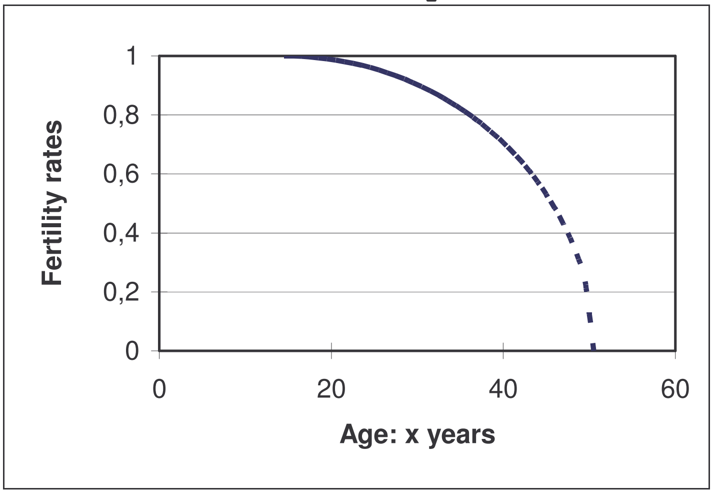
from which it results that a=1, b=V–U and therefore [2] becomes:

$$f^F(X) = \sqrt{1 - \left[(X-U)^2/(V-U)^2\right]}\ . \qquad [4]$$

Tab.1 Values of $f^F(X)$ per age, placed U=14.5 and V=50.5

| Age | $f^F(X)$ | Age | $f^F(X)$ | Age | $f^F(X)$ |
|---|---|---|---|---|---|
| 14.5 | 1.0000 | 26.5 | 0.9428 | 38.5 | 0.7453 |
| 16.5 | 0.9984 | 28.5 | 0.9213 | 40.5 | 0.6916 |
| 18.5 | 0.9938 | 30.5 | 0.8958 | 42.5 | 0.6285 |
| 20.5 | 0.9860 | 32.5 | 0.8660 | 44.5 | 0.5528 |
| 22.5 | 0.9750 | 34.5 | 0.8315 | 46.5 | 0.4581 |
| 24.5 | 0.9606 | 36.5 | 0.7915 | 48.5 | 0.3287 |
| | | | | 50.5 | 0 |

**Fig.1 Trend of the fertility rates from Tab.1**




**The total reproduction R(x), obtained from the initial age up until X-U, and in the absence of female mortality, results from the cumulated fertility rates $f^F(X)$:**

$$\sum_{i=0}^{V-U} f^F(i) = R(V) \quad [5]$$

**which may be written:**

$$\int_0^{X-U} f^F(t)dt = R(X) \quad [6]$$

**the value of which is:**

$$R(X) = (1/2*b)\left[t\sqrt{b^2-t^2} + b^2\, arcsen(t/b)\right]_{0,X-U} \quad [7]$$

**that is:**

$$R(X) = [1/2(V-U)]\cdot\left[(X-U)\sqrt{(V-U)^2-(X-U)^2} + (V-U)^2\, arcsen((X-U)/(V-U))\right]\cdot \quad [8]$$

**For example, fixed x=50.5 we have:**

$$R(50.5) = (1/72)\left[36\sqrt{36^2-36^2} + 36^2\, arcsen(1)\right]\cdot = 28.27$$

children had during the whole of the fertile period.

1.3 As has been previously observed, the physiological procreation of the woman (or the couple) is opposed by a series of restraints which reduces the value of $f^F(X)$.

These restraints will be represented by a curve which, starting from certain values in young age, decreases immediately at the beginning of the fertile period until it reaches a minimum corresponding to the mean age of reproduction when it then rapidily increases.

Indicating with K(x) the system of restraints which develop with age x, we may write:

$$f(X) = f^F(X) - f^F(X).K(X)$$

that is:

$$f(X) = f^F(X).[1 - K(X)] \cdot \qquad [9]$$

For the representative function of K(X) the following expression[1] is proposed:

$$K(X) = e^{-[(V-U)-(X-U)]^B \cdot (X-U)^C / A} \qquad [10]$$

where $U \le X \le (V-U)$

from which[2]:

$$f\ (X) = \sqrt{1 - \left[(X-U)^2 / (V-U)^2\right]}\ \left[1 - e^{-[(V-U)-(X-U)]^B \cdot (X-U)^C / A}\ \right] \qquad [11]$$

1.4 Now several applications are shown which are obtained by adapting function [10] to differentiated fertile rate distributions so as to show the ability of the function in analytically representing the observed data.

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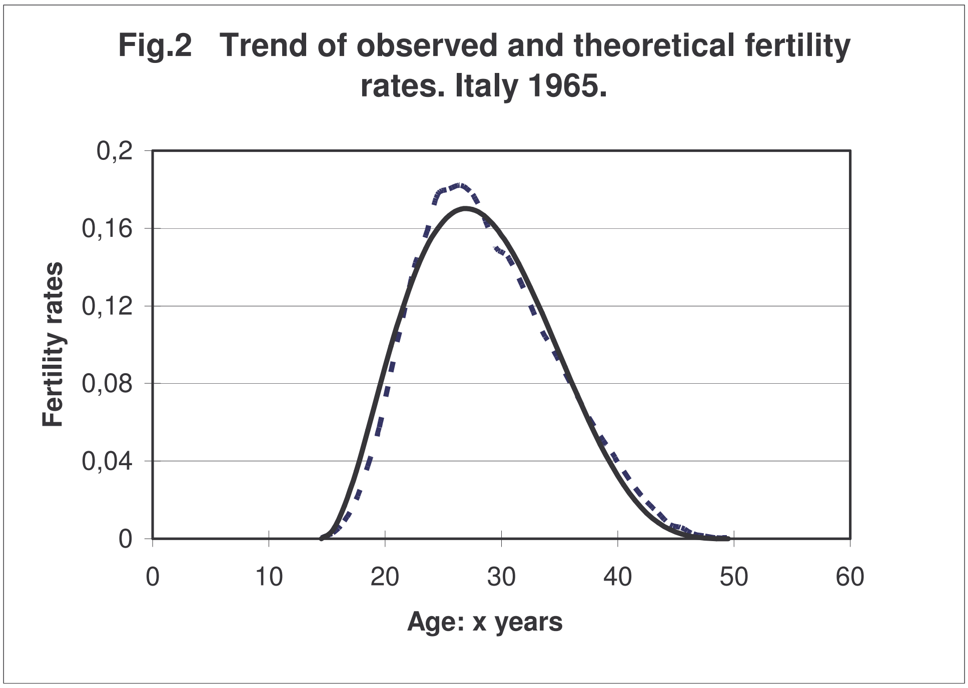
(1) Its minimum value is in age x=(B.U+C.V)/(B+C).

(2) This function is equal to zero for X=U and for X=V. It has positive values for $A > 0$ and for $U \le X \le V$.




The adaptation is carried out by means of a program to be developed on a computer with which, after having determined the temporary parameter values, their definitive values are added after a series of operations which tend to minimize the difference between observed values and theoretical ones.

Tab.2 Comparison between observed values and theoretical ones determined by means of the studied function [10]. Italy 1965.

| Age x | Rates | | Age x | Rates | |
|---|---|---|---|---|---|
| | observed | theoretical | | observed | theoretical |
| 14.5 | 0.00030 | 0.00000 | 32.5 | 0.12050 | 0.12952 |
| 15.5 | 0.00300 | 0.00287 | 33.5 | 0.10690 | 0.11629 |
| 16.5 | 0.00900 | 0.01693 | 34.5 | 0.09670 | 0.10237 |
| 17.5 | 0.02010 | 0.03484 | 35.5 | 0.08520 | 0.08823 |
| 18.5 | 0.03720 | 0.05583 | 36.5 | 0.07370 | 0.07429 |
| 19.5 | 0.05740 | 0.07783 | 37.5 | 0.06260 | 0.06097 |
| 20.5 | 0.08590 | 0.09924 | 38.5 | 0.05290 | 0.04860 |
| 21.5 | 0.11390 | 0.11883 | 39.5 | 0.04420 | 0.03748 |
| 22.5 | 0.13970 | 0.13578 | 40.5 | 0.03470 | 0.02781 |
| 23.5 | 0.15620 | 0.14956 | 41.5 | 0.02620 | 0.01972 |
| 24.5 | 0.17720 | 0.15987 | 42.5 | 0.01890 | 0.01323 |
| 25.5 | 0.18020 | 0.16663 | 43.5 | 0.01310 | 0.00829 |
| 26.5 | 0.18200 | 0.16987 | 44.5 | 0.00730 | 0.00475 |
| 27.5 | 0.17750 | 0.16976 | 45.5 | 0.00500 | 0.00242 |
| 28.5 | 0.16550 | 0.16655 | 46.5 | 0.00240 | 0.00103 |
| 29.6 | 0.15030 | 0.16053 | 47.5 | 0.00130 | 0.00034 |
| 30.5 | 0.14560 | 0.15208 | 48.5 | 0.00050 | 0.00007 |
| 31.5 | 0.13430 | 0.14160 | 49.5 | 0.00030 | 0.00000 |

In Tab.2 and in Fig.2 we report the observed and the theoretical values relative to the fertile rate distribution in Italy in 1965 obtained by applying [11]. The good results of this application may be observed (Tab.3) also as far as the distribution rates of other countries are concerned, chosen differentially, as can be seen from the mean age at childbearing (MA), from the empirical variance (VA), the skewness (SK), and from the total fertility rates (TFR) observed.

Tab.3 Biometric values and matching index between observed and theoretical distributions of various countries and periods referring to not cumulated rates.

| Countries and year of reference | | MA | VA | SK | TFR | $R^2$ | AD |
|---|---|---|---|---|---|---|---|
| Canada | 1925 | 30.124 | 44.812 | 0.0357 | 3.519 | 0.9823 | 0.0777 |
| Colombia | 1964 | 29.647 | 51.579 | 0.0495 | 5.586 | 0.9739 | 0.0876 |
| Congo | 1956 | 27.673 | 51.308 | 0.0770 | 4.745 | 0.9776 | 0.0846 |
| Denmark | 1936 | 28.939 | 40.334 | 0.0570 | 2.137 | 0.9770 | 0.0928 |
| France | 1960 | 28.361 | 35.252 | 0.0989 | 2.699 | 0.9321 | 0.1816 |
| Guatemala | 1970 | 28.798 | 56.767 | 0.0492 | 5.788 | 0.9785 | 0.0705 |
| Italy | 1937 | 30.297 | 41.489 | 0.0374 | 3.030 | 0.9814 | 0.0813 |
| Italy | 1960 | 29.220 | 37.593 | 0.0596 | 2.403 | 0.9812 | 0.0922 |
| Italy | 1965 | 28.706 | 34.794 | 0.0712 | 2.688 | 0.9819 | 0.0895 |
| Italy | 1988 | 28.647 | 29.405 | 0.0506 | 1.340 | 0.9963 | 0.0458 |
| United States | 1970 | 26.070 | 34.826 | 0.0949 | 2.463 | 0.9726 | 0.1064 |
| Zaire | 1961 | 28.715 | 62.856 | 0.0551 | 5.091 | 0.9604 | 0.0551 |

The values of MA, VA, SK and TFR are calculated on the observed data, while by means $R^2$ (Coefficient of determination) and AD (Absolute average differences)[(3)] the degree of approach between observed and theoretical rates is measured.

(3) AD is given by the quotient between the absolute differences among the observed and theoretical values, and the addition of the observed data.