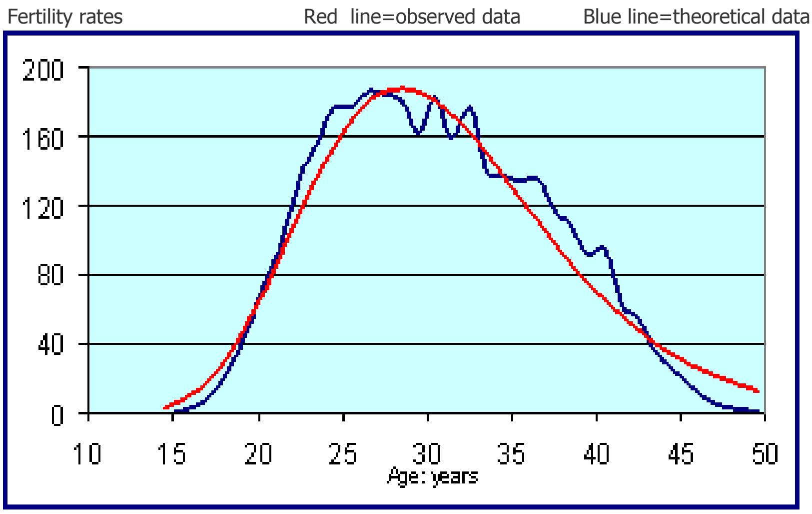
FERTILITY FUNCTIONS

(Source:Petrioli Luciano,"*PRODEMOG 3.0-Demographic software for Windows*", EMMECI-SIENA-ITALY,(2000).

H A D W I G E R



Age-specific fertility rates: Italy, year 1931

1. This function was proposed by Hadwiger in 1940 (see Keyfitz, 1968) with the scope of determining the probability of a girl, born at a given moment, to give birth to a daughter in the interval of age x, x+δx and then to have (always following the female line of descent) a grand-daughter, a great grand-daughter, and so on. Hadwiger represents this probability by means of the density function:

$$f_n(x)dx = \frac{na}{\sqrt{\pi \cdot x^3}} \cdot e^{\left[nac - \left(\frac{n^2a^2}{x} + bx\right)\right]} \cdot dx \qquad [1]$$

where the parameters are *a,b,c* while *n* represents the numerosity of the descendants, that is, if n=1 only one daughter, n=2 with a grand-daughter, n=3 with a great grand-daughter, etc.

Here let us only consider the case in which n=1 and, therefore, that the function of Hadwiger is useful to represent the distribution of the probability (or fertility rates) that a women may have a number of children f(x)dx for age x, x+dx.

Nathan Keyfitz (1968,"Introduction to the mathematics of population", Addison-Wesley, Massachusetts) demonstrates that generally it is:

$$\ln \int_O^{\infty} f_n(x)dx = n\ a(c-2\sqrt{b}) \qquad [2]$$

In this case, namely for n=1, and considering not only female fertility rates, but total rates relative to children of both sexes, indicating the parameters with A, B and C, we have

$$\ln TFT = A(C-2\sqrt{B}) \qquad [3]$$

and hence:

$$C=(\ln TFT)/A + 2\sqrt{B} \qquad [4]$$

2. In place of TFT (Total fertility rate), we can fix a total female fertility rate or the net reproduction rate R_0. Keeping in mind that: MED = $A/\sqrt{B}$; DS= $A/2\sqrt{B^3}$

we have $A = \sqrt{MED^3} / \sqrt{2 \cdot DS};$ $B = MED / 2 \cdot DS$

From these relations we obtain the value of C as seen in (12.24) that is:

$$C = \ln TFT \sqrt{2DS / MED^3} + \sqrt{2MED / DS}$$

It is easy to demonstrate that the Inverse-Gaussian (GI) and the Hadwiger (HW) give, in substance, the same results. In fact, by equaling the variance of GI and HW, see (12.9) and the values of MED and DS respectively, indicated above, we have:

$$M=A/\sqrt{B} \quad ; \quad N=2A^2$$

From these relations it is seen that if the parameters A and B of the studied distribution are noted, calculated by means of HW, we can continue with the determination of the corresponding parameters of GI, considering the case in which the variable of transformation is zero.

The relation between the two functions is valid, nonetheless, even when for each of these we would have introduced several variables of transformation (TV or V) which in the table, for example, are set at zero, five, ten years. In this case, with TV for the Hadwiger function and V for the Inverse-Gaussian, from which let us write D=TV.V, we have:

$$M=(A/\sqrt{B})+D; \quad N = [\ 2\left(A+D\sqrt{B}\right)^3\ ] / A$$

Mean and variance are reported, respectively, in the Table 1.

Table 1 Parameters of the functions of Hadwiger and of the Inverse-Gaussian applied to the distribution of fertility rates for Togo-1961.

Transformation of the variable	Initial parameters		Final parameters			Mean	Empirical variance
	H A D W I G E R						
	A	B	A	B	C		
TV							
0	14.49892	0.24264	12.68749	0.17975	1.00233	29.434	60.652
5	10.96610	0.20143	9.53190	0.14124	0.95700	"	"
	I N V E R S E - G A U S S I A N						
	N	M	N	M	C	Mean	Empirical variance
V							
0	420.4371	29.434	322.4291	30.0847	7.2994	29.434	60.652
5	240.5119	24.434	179.1312	25.5175	7.3139	"	"
10	121.0149	19.434	86.0956	21.4834	7.3690	"	"

The initial parameters of the two functions, as it is easy to prove, maintain the relationships established in this note, demonstrating the equality between the GI and the HW, as previously seen. But if we use those initial values in the iterative process, we see that the final parameters of the two functions lose, in part, this relation and the adjustment to the observed data occurs in a differentiated manner. The same observation is also true for the relation seen between Gompertz and Gumbel.

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